On the mathematical theory of superfluidity

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Abstract

We provide a general operator algebraic formulation of the theory of superfluidity in Bose systems, with the aim of investigating the relationships of this phenomenon both to off-diagonal long range order (ODLRO) and to a mathematically precise version of Landau's picture of elementary excitations. Our principal results are that ODLRO leads both to rotational superfluidity and to Goldstone excitations, while the neo-Landau picture accounts for the translational superfluidity of flow along a pipe. The latter picture is realized by the Lieb–Liniger–Girardeau model. Open problems are briefly discussed.

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1. Introduction

Superfluidity, or frictionless flow, is a quantum phenomenon on the macroscopic scale, which arises in ordered phases of superconductors [1], liquid He II [2, 3] and trapped Bose gases [4]. The quantum theory of these phases is not only intrinsically fascinating but is also remarkable for its ramifications in other areas, such as the Higgs [5] mechanism in particle physics and the ordered structures, far from equilibrium, of laser light [6] and biocells [7]. Although there is an enormous literature on the subject, it is probably fair to say that there are two principal lines along which theories of superfluidity have developed over the years.

The first of these is centred on the phenomenon of Bose–Einstein (BE) condensation. Specifically, Tisza [8] and London [2] proposed that the superfluid phase of He II is characterized by a version of this condensation that prevails even in interacting Bose systems. The precise form of this condensation was subsequently formulated by Penrose and Onsager [9] and later extended to superconductors by Yang [10], who coined the term *off-diagonal long range order* (ODLRO) to describe it. In fact the characterization of superconductivity by ODLRO encapsulated the essential content of the Bardeen–Cooper–Schrieffer (BCS) theory [11]. In general, the ODLRO condition is expressed in terms of a *macroscopic wavefunction*.

For a system in a pure phase, this is just the position-dependent expectation value of a quantum field[3], which in the bosonic case is just that representing its particles, while in the fermionic case it is the product of the fields representing its particles of opposite spin. Thus, in either case, it corresponds to a spontaneous *gauge symmetry breakdown*.

The second line of development of the theory of superfluidity stems from Landau's [12] picture of the low-lying excitations of He II. This was based on a remarkable combination of quantum theory and hydrodynamics which depicted these excitations as phonons and rotons, corresponding to quantized sound waves and vortices, respectively.

Although these two approaches to superfluidity are based on different ideas, they have been brought together, to some extent, by the works of Bogoliubov [13][4] and Feynman [15] on interacting Bose systems (weakly interacting in Bogoliubov's case). In fact, both of these works supported Landau's picture and also satisfied the ODLRO condition.

Further, both approaches led to models wherein a superfluid comprises two spatially coexisting fluids, the hydrodynamics of the first being irrotational and frictionless and that of the second being normal: thus the superfluid proper is just the first component. According to the BE condensation picture, this corresponds to the condensate, while in the Landau picture it corresponds to the residual ground-state component of the mixed state of the system. In both pictures, the normal fluid component is carried by the excitations. At a phenomenological level, the two-fluid model has enjoyed great success in accounting for the observed thermodynamical properties of superfluids [1–3, 12].

As regards the problem of superfluidity *per se*, the situation is less clear, at least on the level of mathematical physics. In the BE condensation, or ODLRO, picture, it depends on the *assumption* that the condensate flow is irrotational and frictionless, with velocity potential given by the phase angle of the macroscopic wavefunction. In Landau's picture, the frictionlessness of the flow of He II at sufficiently small velocities stems from the stability of its ground state against the generation of quasi-particle excitations, i.e. phonons and rotons. However, that picture carries no considerations of possible instabilities against more complicated excitations. In fact, it appears to us that the existing rigorous results about superfluidity are confined to proofs that it prevails in (a) superconductors, subject to the assumptions of ODLRO, gauge covariance and thermodynamical stability [16], and (b) trapped, dilute, interacting Bose gases [4, 17, 18].

The object of the present paper is to make a general analysis of the phenomenon of superfluidity of suitably interacting bosonic systems, with the aim of obtaining sharp mathematical criteria for its occurrence. Here we emphasize that there are two different versions of this phenomenon. The first, which we shall call the *translational* one, concerns the frictionless flow of the superfluid along a cylindrical pipe; the second, which we shall call the *rotational* version, is the phenomenon whereby a superfluid in a rotating drum remains at rest and thus does not contribute to the moment of inertia of the drum-plus-fluid. In fact, we find that the conditions for these two manifestations of superfluidity are quite different from one another. The rotational kind will be shown to stem from a superselection rule associated with ODLRO, similar to that governing persistent currents induced by a trapped magnetic field in a superconducting ring. The translational kind, on the other hand, will be shown to be more complicated, in that further conditions need to be added to Landau's quasi-particle picture in order to ensure the metastability of frictionless flow along a pipe.

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[3] This description is appropriate if one describes the system in terms of its field algebra. While it is convenient to adopt this description, it is not really necessary, since one can equally well formulate ODLRO in terms of the observable algebra, given by the globally gauge invariant subalgebra of the field algebra.

[4] See also the excellent recent review of Bogoliubov's work by Zagrebnov and Bru [14].

We formulate our treatment of superfluidity within the operator algebraic framework of quantum statistical physics, which is natural for the study of intrinsic properties of matter in the thermodynamic limit [19–22]. Thus we start, in section 2, with a concise pedagogical formulation, within this framework, of the generic model of interacting Bose systems on which our treatment of superfluidity will be based; and in section 3 we briefly review the conditions for spontaneous Galilei and gauge symmetry breakdown, with associated Goldstone bosonic excitations, in this model. We then pass on, in section 4, to a treatment of rotational superfluidity, showing that this stems from a superselection rule associated with ODLRO: this includes a derivation of the Onsager–Feynman [15, 23] quantization rule for vortices in superfluids. Section 5 is devoted to a treatment of the problem of translational superfluidity. There we show that the frictionless flow along a pipe cannot be stable against *all* localized modifications of state. Evidently, this raises questions about the character of the metastability of this flow, and these are addressed in section 6, where a refined version of Landau's picture is formulated in terms of the elementary excitations in superfluids and explicitly realized by the Lieb–Liniger–Girardeau model [24, 25]. We conclude in section 7 with a discussion of both the results and the open questions concerning the stability or metastability of superfluid states.

Throughout the paper we shall employ units in which $\hbar$, $k_{\text{Boltzmann}}$ and the mass per particle are unity.

2. The model

We take our model to be an infinitely extended system, Σ, of bosons of one species that occupies a d-dimensional Euclidean space, X, and we formulate this model in standard operator algebraic terms (cf [19–22]).

The field algebra. This is a $C^\star$-algebra of the quantized field representing the particles of Σ. It is constructed as an algebra of operators in the Fock–Hilbert space, $\mathcal{H}_0$, which together with its Fock vacuum vector, Φ_0, is defined by the following conditions:

(i) There is a map W, the Weyl map, of $L^2(X)$ into the unitaries in $\mathcal{H}_0$ that satisfies the canonical commutation relations (CCR)

$$W(f)W(g) = W(f + g)\exp(\mathrm{i}Im(f, g)) \,\forall f, \ g \in L^2(X), \tag{2.1}$$

where $(\cdot, \cdot)$ is the $L^2(X)$ inner product.
(ii) Φ_0 is cyclic with respect to the algebra of the polynomials of these operators.
(iii) The Fock vacuum state $\phi_0 := (\Phi_0, \cdot\Phi_0)$ is given by the formula

$$\phi_0(W(f)) = (\Phi_0, W(f)\Phi_0) = \exp\left(-\tfrac{1}{2}\|f\|^2\right) \forall f \in L^2(X). \tag{2.2}$$

It then follows from (i)–(iii) (cf [20, chapter 3]) that $W(f)$ takes the form

$$W(f) = \exp[\mathrm{i}(\psi(f) + \psi(f)^\star)], \tag{2.3}$$

where $\psi(f)$ and $\psi(f)^\star$ are closed densely defined operators in $\mathcal{H}_0$ with the properties that (a) $\psi(f)$ annihilates Φ_0, (b) Φ_0 is cyclic with respect to the algebra of the polynomials in $\{\psi(f)^\star | f \in L^2(X)\}$, and (c) ψ and $\psi^\star$ satisfy the following form of the CCR:

$$[\psi(f), \psi(g)^\star] = (g, f). \tag{2.4}$$

Thus, $\psi(f)$ is a smeared version, $\int_X \mathrm{d}x\psi(x)f(x)$, of a quantum field $\psi(x)$, that satisfies the formal CCR

$$[\psi(x), \psi^\star(y)] = \delta(x - y); [\psi(x), \psi(y)] = 0. \tag{2.4}'$$

In order to describe the local properties of the field, we introduce the set, $\mathcal{L}$, of bounded open subsets of X and, for each Λ in $\mathcal{L}$, we define $\mathcal{H}_\Lambda$ to be the subspace of $\mathcal{H}_0$ generated by the application to Φ_0 of the polynomials in $\{\psi(f)^\star | f \in L^2(\Lambda)\}$ and $\mathcal{F}_\Lambda$ to be the $W^\star$-algebra of bounded operators in this space. Thus, the local algebras $\{\mathcal{F}_\Lambda | \Lambda \in \mathcal{L}\}$ satisfy the conditions of isotony and local commutativity, namely

$$\mathcal{F}_\Lambda \subset \mathcal{F}_{\Lambda'} \qquad \text{if} \quad \Lambda \subset \Lambda'$$

and

$$[A, B] = 0 \, \forall \, A \in \mathcal{F}_\Lambda, \; B \in \mathcal{F}_{\Lambda'} \qquad \text{if} \quad \Lambda \cap \Lambda' = \emptyset,$$

respectively. In view of the isotony property, $\mathcal{F}_\mathcal{L} := \bigcup_{\Lambda \in \mathcal{L}} \mathcal{F}_\Lambda$ is a $\star$-algebra. Equipped with the $\mathcal{H}_0$ operator norm, it becomes a normed $\star$-algebra, whose completion, $\mathcal{F}$, is a $C^\star$-algebra, termed the quasi-local field algebra of the model.

We define γ, σ and ξ to be the representations in $Aut(\mathcal{F})$ of the additive groups $S^{(1)}$, X and X, corresponding to gauge transformations, space translations and Galilei velocity boosts, respectively, by the formulae

$$\gamma(\theta)[W(f)] = W(f \exp(i\theta)) \, \forall \, \theta \in \mathbf{R}(\mathrm{mod}2\pi), \tag{2.5}$$

$$\sigma(x)[W(f)] = W(f_x) \, \forall \, x \in X, \qquad \text{with} \qquad f_x(y) = f(y - x) \tag{2.6}$$

and

$$\xi(v)[W(f)] = W(f_v) \, \forall \, v \in X, \qquad \text{with} \qquad f_v(x) = f(x)\exp(iv \cdot x); \tag{2.7}$$

or, formally,

$$\gamma(\theta)[\psi(x)] = \psi(x)\exp(i\theta), \qquad \sigma(x)[\psi(y)] = \psi(x + y) \qquad \text{and}$$
$$\xi(v)[\psi(x)] = \psi(x)\exp(iv \cdot x). \tag{2.8}$$

The algebra of observables and its automorphisms. Since observables are gauge invariant quantities, we take the $C^\star$-algebra, $\mathcal{A}$, of quasi-local bounded observables of Σ to be the $C^\star$-subalgebra of $\mathcal{F}$ comprising its gauge invariant elements. Likewise, we define $\mathcal{A}_\Lambda$ and $\mathcal{A}_\mathcal{L}$ to be the algebras comprising the gauge invariant elements of $\mathcal{F}_\Lambda$ and $\mathcal{F}_\mathcal{L}$, respectively.

Since, by equations (2.5)–(2.7), $\gamma(\theta)$ commutes with both $\sigma(x)$ and $\xi(v)$, it follows that $\mathcal{A}$ is stable under the latter two automorphisms. Consequently, their restrictions to $\mathcal{A}$ are automorphims of this algebra, and so σ and ξ operate as representations of space translations and Galilei boosts, respectively, in $Aut(\mathcal{A})$, as well as in $Aut(\mathcal{F})$.

Local normality and unbounded observables. We assume that, of the myriad mathematical states and representations of $\mathcal{A}$, the physical ones are locally normal, i.e. that their restrictions to the local algebras $\mathcal{A}_\Lambda$ are normal, since this is precisely the condition that, with probability 1, they do not admit an infinite number of particles in any bounded region [26].

Furthermore, the locally normal representations of $\mathcal{A}$, and also of $\mathcal{F}$, are those that reduce to the Fock representation in any bounded spatial region [26]. This ensures that the formulation of the bounded local observables can be canonically extended to the unbounded ones [27]. Specifically, those that are localized within Λ are represented by the unbounded self-adjoint operators affiliated to $\mathcal{A}_\Lambda$, i.e. by those that commute with the commutant of this algebra in $\mathcal{A}$. In particular, the observables N_Λ, J_Λ and T_Λ, representing the number of particles, the current and the kinetic energy, respectively, in Λ are given in terms of a differentiable orthonormal

basis, $\{f_r\}$, of $L^2(\Lambda)$ by the equations (cf [28])

$$N_\Lambda = \sum_r \psi(f_r)^\star \psi(f_r), \qquad J_\Lambda = \frac{i}{2} \sum_r (\psi(f_r)^\star \psi(\nabla f_r) - \psi(\nabla f_r)^\star \psi(f_r)),$$

$$T_\Lambda = \frac{1}{2} \sum_r \psi(\nabla f_r)^\star \cdot \psi(\nabla f_r) \tag{2.9}$$

or formally,

$$N_\Lambda = \int_\Lambda dx\, n(x) : J_\Lambda = \int_\Lambda dx\, j(x), \qquad T_\Lambda = \int_\Lambda dx\, t(x), \tag{2.9$'$}$$

where

$$\begin{aligned}
n(x) &= \psi^\star(x)\psi(x), \\
j(x) &= -\frac{i}{2}(\psi^\star(x)\nabla\psi(x) - [\nabla\psi^\star(x)]\psi(x)), \\
t(x) &= \frac{1}{2}\nabla\psi^\star(x)\cdot\nabla\psi(x)
\end{aligned} \tag{2.9$''$}$$

and Neumann boundary conditions are employed (cf [28]).

Physical states, representations and dynamics. The condition of local normality does not suffice for the characterization of physical states and representations, as there are locally normal ones that do not support a $C^\star$-dynamics of Σ, as given by a natural limit of that of a finite version of this model [29–31]. Instead, the system admits just a $W^\star$-dynamics via such a limit, and this is confined to certain privileged representations, π, of $\mathcal{A}$ [31]. To be specific, the dynamics in an admissible representation π corresponds to a one-parameter group, $\{\alpha(t)|t \in \mathbf{R}\}$, of automorphisms of $\pi(\mathcal{A})''$, defined according to the following prescription [31]. For $\Lambda \in \mathcal{L}$, we assume that the Hamiltonian operator of the finite version, Σ_Λ, of Σ located in Λ is an affiliate, H_Λ, of $\mathcal{A}_\Lambda$. The evolute of an observable $A(\in \mathcal{A}_\Lambda)$ of Σ_Λ at time t is then $A_\Lambda(t) := (\exp(iH_\Lambda t)A\exp(-iH_\Lambda t)$ and our condition for a representation π to be physical is that, for any $A \in \mathcal{A}_\mathcal{L}$, $\pi(A_\Lambda(t))$ converges strongly to a limit, necessarily in $\pi(\mathcal{A})''$, as a Λ increases to X in, say, Fisher's [32] sense. This limiting procedure then serves to define the strongly continuous dynamical group $\alpha(\mathbf{R})(\in Aut(\pi(\mathcal{A})'')$ by the formula

$$\alpha(t)[\pi(A)] = s - \lim_{\Lambda\uparrow X} \pi(\exp(iH_\Lambda t)A\exp(-iH_\Lambda t)) \,\forall\, A \in \mathcal{A}, \qquad t \in \mathbf{R}. \tag{2.10}$$

To lighten the notation, we define

$$\tilde{A} := \pi(A) \qquad \text{and} \qquad \tilde{A}_t := \alpha(t)[\pi(A)] \,\forall\, A \in \mathcal{A}, \qquad t \in \mathbf{R}. \tag{2.11}$$

We generally assume that H_Λ takes the form

$$H_\Lambda = \frac{1}{2}\int_\Lambda dx\,\nabla\psi^*(x)\cdot\nabla\psi(x) + \int_{\Lambda^2} dx\,dy\,\psi^*(x)\psi^*(y)V(x-y)\psi(y)\psi(x) \tag{2.12}$$

We now assume that the physical representations of $\mathcal{A}$ are the locally normal ones that support the dynamics given by this prescription. Correspondingly, we take the physical states of the model to be those whose GNS representations satisfy this physicality condition. Denoting the GNS triple of such a state ϕ by $(\mathcal{H}, \pi, \Phi)$, we define $\tilde{\phi}$ to be the canonical extension of ϕ to $\pi(\mathcal{A})''$ according to the formula

$$\tilde{\phi}(M) = (\Phi, M\Phi) \,\forall\, M \in \pi(\mathcal{A})''. \tag{2.13}$$

The state ϕ is then termed stationary if $\tilde{\phi}$ is invariant under $\alpha(\mathbf{R})$, in which case these automorphisms are unitarily implemented by a strongly continuous representation U of $\mathbf{R}$ in $\mathcal{H}$, as defined by the formula [33]

$$U(t)\tilde{A}\Phi = \tilde{A}_t\Phi \,\forall\, A \in \mathcal{A}, \qquad t \in \mathbf{R}. \tag{2.14}$$

We denote by H the Hamiltonian operator given by $-$ i time the generator of the group $U(\mathbf{R})$, i.e.

$$U(t) = \exp(iHt) \,\forall\, t \in \mathbf{R}. \tag{2.15}$$

Likewise, if ϕ is translationally invariant, the automorphisms $\sigma(X)$ are unitarily implemented by a representation S of X in $\mathcal{H}$, as defined by the formula

$$S(x)\pi(A)\Phi = \pi(\sigma(x)A)\Phi \,\forall\, A \in \mathcal{A}, \qquad x \in X \tag{2.16}$$

and, assuming that $S(x)$ is continuous in x, we denote by P the momentum operator given by $-$ i time the generator of the group $S(X)$, i.e.

$$S(x) = \exp(iP \cdot x) \,\forall\, x \in X. \tag{2.17}$$

Note. This formulation of the states and dynamical automorphisms of the observable algebra $\mathcal{A}$ may readily employed for that of the states and dynamics of the field algebra $\mathcal{F}$. We note, in particular, that a state ϕ on $\mathcal{A}$ has a unique extension to a gauge invariant state $\phi_{\mathcal{F}}$ on $\mathcal{F}$, defined by the formula

$$\phi_{\mathcal{F}}(F) = \phi(pF), \qquad \text{where} \quad pF = (2\pi)^{-1} \int_0^{2\pi} d\theta \, \gamma(\theta) F \,\forall\, F \in \mathcal{F}. \tag{2.18}$$

The KMS thermal equilibrium condition. As first proposed by Haag *et al* [34], the equilibrium states of a system are those that satisfy the Kubo–Martin–Schwinger (KMS) condition. In the present setting, this condition, as applied to the state ϕ of Σ at inverse temperature β, takes the following form. For any pair of elements A, B of $\mathcal{A}$, there is a function F_{AB} on the strip $\text{Im}(z) \in [0, \beta]$ of the complex plane, that is analytic in its interior and continuous on its boundaries, where it reduces to the forms

$$F_{AB}(t + i\beta) = \tilde{\phi}(\tilde{A}_t \tilde{B}) \qquad \text{and} \qquad F_{AB}(t) = \tilde{\phi}(\tilde{B}\tilde{A}_t) \,\forall\, t \in \mathbf{R}, \tag{2.19}$$

or, formally,

$$\tilde{\phi}(\tilde{A}_t \tilde{B}) = \tilde{\phi}(\tilde{B}\tilde{A}_{t+i\beta}) \,\forall\, A, \; B \in \mathcal{A}, \quad t \in \mathbf{R}. \tag{2.19}'$$

Support for the general characterization of thermal equilibrium by the KMS condition is provided by the fact that

(i) it implies the stationarity of the state concerned [35];
(ii) it is satisfied by the canonical equilibrium state of a finite system and by any infinite volume limit thereof;
(iii) it is equivalent to the various dynamical and thermodynamical equilibrium properties that are natural desiderata of equilibrium states [36–39].

The set, $\mathcal{S}_{\beta}$, of KMS states on $\mathcal{A}$ at inverse temperature β is manifestly convex and, as proved by Emch *et al* [35], its extremal elements are primary and enjoy the essential properties of pure thermodynamical phases. Thus, the central decomposition of a KMS state serves precisely to resolve it into pure equilibrium phases.

Ground-state condition. The condition for ϕ to be a ground state of Σ is that of KMS corresponding to $\beta = \infty$ and reduces to the condition that the Fourier transform of the function, or distribution, $t \to \tilde{\phi}(\tilde{B}\tilde{A}_t)$ has support in $\mathbf{R}_+$. Equivalently, by equations (2.14) and (2.15), it is equivalent to the condition that H is a positive operator that annihilates the vector Φ.

Advent of the chemical potential. A remarkable work by Araki *et al* [40] established that the thermodynamical parameter known as the chemical potential emerges from the structures of

the algebras $\mathcal{A}$ and $\mathcal{F}$ and the KMS condition. Specifically, it established that, if ϕ is a primary element of $\mathcal{S}_\beta$, then its canonical gauge invariant extension to $\mathcal{F}$ satisfies the corresponding KMS condition with respect to the dynamical automorphism group $\alpha_\mu(\mathbf{R})$, defined by the formula

$$\alpha_\mu(t) = \alpha(t)\gamma(-\mu t), \tag{2.20}$$

where μ is a real-valued parameter. Since, by equations (2.5) and (2.9),

$$\gamma(\theta)A = \exp(iN_\Lambda\theta)A\exp(-iN_\Lambda\theta) \,\forall\, A \in \mathcal{A}_\Lambda, \qquad \theta \in [0, 2\pi], \quad \Lambda \in \mathcal{L}, \tag{2.21}$$

it follows from equations (2.10) and (2.20) that the modification of $\alpha(t)$ by the factor $\gamma(-\mu t)$ corresponds to the addition of the term $-\mu N_\Lambda$ to the local Hamiltonian H_Λ and thus to the advent of a chemical potential of value μ.

Off-diagonal long range order. The concept of long range order (ODLRO) is a generalization to interacting systems of that of Bose–Einstein condensation. Specifically, the state ϕ is said to possess ODLRO [9, 10] if there is a complex classical field Ψ that does not tend to zero at infinity and that satisfies the condition

$$\lim_{|y|\to\infty} [\langle\phi; \psi^\star(x)\psi(x' + y)\rangle - \overline{\Psi}(x)\Psi(x' + y)] = 0. \tag{2.22}$$

In fact, this condition defines Ψ up to a constant phase factor[5]. This field is termed the *macroscopic wavefunction* and represents the condensate. Correspondingly, the condensate density, $\rho_c(x)$, and the associated current density, $j_c(x)$, are defined by the formulae

$$\rho_c(x) = \overline{\Psi}(x)\Psi(x) \tag{2.23}$$

and

$$j_c(x) = -\frac{i}{2}(\overline{\Psi}(x)\nabla\Psi(x) - \Psi(x)\nabla\overline{\Psi}(x)). \tag{2.24}$$

We shall presently discuss the significance of ρ_c and j_c, with respect to the phenomenon of superfluidity.

General phenomenological picture: the two-fluid model. According to an empirically based hydrodynamical picture [2, 3, 12], a fluid in its condensed, superfluid phase behaves as a mixture of two fluids, one of which flows frictionlessly and irrotationally, while the other enjoys normal classical viscous properties. These are termed the superfluid and normal components, respectively, of the fluid. Thus the position-dependent hydrodynamical density, $\rho(x)$, and current-density, $j(x)$, take the forms

$$\rho(x) = \rho_s(x) + \rho_n(x) \tag{2.25}$$

and

$$j(x) = \rho_s(x)v_s(x) + \rho_n(x)v_n(x), \tag{2.26}$$

where ρ_s and v_s are the density and drift velocity, respectively, of the superfluid component, while ρ_n and v_n are those of the normal component. Further, the irrotationality condition for the superfluid component is simply that

$$\mathrm{curl}\, v_s(x) = 0. \tag{2.27}$$

Relationship between condensate and superfluid densities. The available results on the relationship between the condensate density, ρ_c and the superfluid density, ρ_s, are those

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[5] The proof of this assertion for pair fields in the fermion case [16, proposition 3.1] carries through trivially for the present bosonic one.

of Penrose and Onsager [9] for He II and of Lieb *et al* [18] for the trapped dilute Bose gas, discussed in the paragraph following equation (2.28). In the former case, it is argued that ρ_c is approximately 8% of ρ_s; while in the latter one, it is proved that $\rho_s = \rho_c$ and that, at least in the case of uniform drift velocity, the condensate and superfluid current densities are equal.

In view of these results, it is tempting to conjecture that, in general, ρ_s and $\rho_s v_s$ are simply proportional to ρ_c and j_c, respectively, with the same constant of proportionality. Assuming this to be the case, it follows from equations (2.23) and (2.24) that

$$v_s = \nabla(\arg(\Psi)), \tag{2.28}$$

and hence that the irrotationality condition (2.27) is satisfied.

Note on the Lieb–Seiringer–Yngvason (LSY) model [4, 18]. This represents a trapped dilute Bose gas of N particles in a three-dimensional cube of side L with periodic boundary conditions. The particles are assumed to interact via a two-body potential V, whose scattering length is a and the diluteness of the gas is represented by the condition that Na/L remains fixed and finite in the limit $N \to \infty$. Thus the LSY model, as treated in this limit, is an infinite system, though one that is quite different from the model Σ that we have just formulated. Lieb and Seiringer [17] have established that the ground state of the LSY model exhibits ODLRO and that its macroscopic wavefunction Ψ is determined by a variational principle, proposed by Gross [41] and Pitaevski [42]. Moreover, this result prevails in the situation where the gas is subjected to an external, one-body potential [43].

3. Breakdown of Galilei and Gauge symmetries

Let us first consider the action of the Galilei boost $\xi(v)$ on the local current J_Λ. By equations (2.8), (2.9)$'$ and (2.9)$''$,

$$\xi(v)J_\Lambda = J_\Lambda + N_\Lambda v, \tag{3.1}$$

which implies that if $\phi(N_\Lambda) \neq 0$, then the state ϕ is not Galilei invariant. In other words, except in the trivial case when ϕ is the Fock vacuum, this state breaks the Galilei symmetry and hence [44] it supports excitations of Goldstone bosons corresponding to quantized density waves, i.e. phonons[6].

A different kind of symmetry breakdown and associated Goldstone bosons arises in the case where ϕ satisfies both the KMS and ODLRO conditions. The argument runs as follows. Since ϕ is a primary KMS state, it follows [40] that its gauge invariant extension $\phi_{\mathcal{F}}$ to the field algebra $\mathcal{F}$ satisfies the KMS condition w.r.t. the automorphisms $\alpha_\mu(\mathbf{R})$ defined by equation (2.20). Furthermore [35], the central decomposition of $\phi_{\mathcal{F}}$ resolves this state into primary α_μ-KMS states on $\mathcal{F}$ and thus takes the form

$$\phi_{\mathcal{F}} = \int_{\mathcal{P}} \nu \, \mathrm{d}P(\nu), \tag{3.2}$$

where P is a probability measure on this set of primaries, which we denote by $\mathcal{P}$. Hence, as ϕ is the restriction of $\phi_{\mathcal{F}}$ to $\mathcal{A}$,

$$\phi = \int_{\mathcal{P}} \nu_{|\mathcal{A}} \, \mathrm{d}P(\nu). \tag{3.3}$$

Moreover, since $\mathcal{A}$ is the gauge invariant subalgebra of $\mathcal{F}$, it follows from equation (2.20) that the automorphisms $\alpha_\mu(t)$ reduce to $\alpha(t)$ on $\mathcal{A}$ and hence that the states $\nu_{|\mathcal{A}}$ are KMS states of

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[6] Strictly speaking, it is only at a formal level that this has been demonstrated for continuous systems in [44, 45].

Σ for ν a.e. w.r.t. P. Therefore as ϕ is a primary and hence [35] an extremal KMS state on $\mathcal{A}$, it follows from equation (3.3) that

$$\nu_{|\mathcal{A}} = \phi \qquad \text{for} \quad \nu \text{ a.e. w.r.t } P. \tag{3.4}$$

Consequently, as $\psi^\star(x)\psi(x' + y)$ is affiliated to $\mathcal{A}$,

$$\langle \phi; \psi^\star(x)\psi(x' + y) \rangle = \langle \nu; \psi^\star(x)\psi(x' + y) \rangle \qquad \text{for} \quad \nu \text{ a.e. w.r.t. } P. \tag{3.5}$$

Moreover, since ν is primary and therefore strongly clustering [35, 46],

$$\lim_{y \to \infty} [\langle \nu; \psi^\star(x)\psi(x' + y) \rangle - \overline{\langle \nu; \psi(x) \rangle} \langle \nu; \psi(x' + y) \rangle] = 0$$

and consequently by equation (3.5)

$$\lim_{y \to \infty} [\langle \phi; \psi^\star(x)\psi(x' + y) \rangle - \overline{\langle \nu; \psi(x) \rangle} \langle \nu; \psi(x' + y) \rangle] = 0 \qquad \text{for} \quad \nu \text{ a.e. w.r.t. } P. \tag{3.6}$$

On comparing this equation with the ODLRO condition (2.22) and recalling that the latter formula defines the macroscopic wavefunction Ψ up to a constant phase angle, it follows that

$$\langle \nu; \psi(x) \rangle = \Psi(x) \exp(ic_\nu) \qquad \text{for} \quad \nu \text{ a.e. w.r.t. } P, \tag{3.7}$$

where c_ν is a real-valued constant. Since, by equation (2.8), $\gamma(\theta)[\psi(x)] = \psi(x) \exp(i\theta)$, it follows that the components ν of $\phi_\mathcal{F}$ break the gauge symmetry. This implies rigorous bounds on the rate of clustering [47].

This argument does not apply to the zero temperature situation for two reasons. Firstly, the work of Araki *et al* [40] concerning the extension of a primary KMS state ϕ on $\mathcal{A}$ to an α_μ-KMS state on $\mathcal{F}$ is not applicable here, since ground states do not breed the modular automorphisms on which the argument depends. Secondly, the argument of Emch *et al* [35] concerning the central decomposition of KMS states into extremal KMS states depends on the finiteness of the inverse temperature β and so is not applicable to ground states.

In view of this situation, we relate ODLRO to symmetry breakdown in ground states by a strategy different from that employed above for thermal states. Specifically, instead of deriving the symmetry breakdown from ODLRO, we proceed in the opposite, and more usual, direction (cf [45, 48]). Thus, we assume that the model supports primary ground states ν on $\mathcal{F}$ with respect to the automorphisms $\alpha_\mu(\mathbf{R})$ that fulfil the gauge symmetry breaking condition analogous to equation (3.7), i.e.

$$\langle \nu; \psi(x) \rangle = \Psi_1(x), \tag{3.8}$$

where Ψ_1 is a classical field that does not tend to zero at infinity. It then follows from the clustering property of primary states that

$$\lim_{|y| \to \infty} [\langle \nu; \psi^\star(x)\psi(x + y) \rangle - \overline{\Psi}_1(x)\Psi_1(x + y)] = 0.$$

Hence, as $\psi^\star(x)\psi(x + y)$ is affiliated to $\mathcal{A}$, the restriction ϕ of ν to this algebra satisfies the ODLRO condition

$$\lim_{|y| \to \infty} [\langle \phi; \psi^\star(x)\psi(x + y) \rangle - \overline{\Psi}_1(x)\Psi_1(x + y)] = 0.$$

Further, in view its gauge covariance, the model must support different symmetry-breaking states ν on $\mathcal{F}$ that modify the function Ψ_1 by constant phase factors running from 0 to 2π. Thus, the assumption of condition (3.8) leads to the same picture of the connections between ODLRO, symmetry breakdown and quasi-particle excitations for ground states as for thermal ones. However, this leaves open the question of whether Σ supports translationally invariant ground states on $\mathcal{A}$ that possess ODLRO but do not extend to gauge symmetry breaking ground states on $\mathcal{F}$.

Assuming, however, that the ground state ϕ exhibits ODLRO and extends to gauge symmetry breaking ground states, w.r.t. $\alpha_\mu(\mathbf{R})$, on $\mathcal{F}$, we may infer [45, 48] from the above argument that $\phi_{\mathcal{F}}$ supports Goldstone bosonic excitations corresponding to the action of the field ψ or $\psi^\star$ on the cyclic vector Φ. These are evidently single particle excitations and are realized in Bogoliubov's treatment of weakly interacting Bose gases [13]. Assuming that the state ϕ is translationally invariant, as in a fluid rather than a crystalline phase, these excitations are naturally described in terms of the Fourier transform $a(k)$ of $\psi(x)$ and the states of Σ carry them take the form $\phi(a(k)^\star(\cdot)a(k))$ and $\phi(a(k)(\cdot)a(k)^\star)$. We note here that although these excitations look quite different from the phonons carried by the Galilei symmetry breakdown, there are heuristic indications [49, 50], of a spectral nature, that they are closely related to them. We also remark that Lieb *et al* [18] have employed a different definition of spontaneous symmetry breakdown, expressed in terms of Bogoliubov quasi-averages [13, 14]; and on that basis they have established the spontaneous breakdown of gauge symmetry in the ground state of their model of a dilute, trapped Bose gas, which we discussed briefly at the end of section 2. It is an open problem whether and how their picture of symmety breakdown is related to that presented above.

4. ODLRO and rotational superfluidity

We base our treatment of rotational superfluidity on the picture of Σ relative to a frame of reference that rotates with uniform angular velocity about an axis Oz, say. This is the picture of Σ in a rotating drum, as viewed from a frame of reference in which the drum is at rest and in the idealization wherein the boundaries of the drum are at infinity. We assume that the system is subjected to a conservative external field that stabilizes it against the centrifugal force, even in the limit where the drum becomes infinite.

Thus, the finite volume Hamiltonians, $\{H_\Lambda'\}$, which govern the dynamics of Σ relative to the rotating reference frame according to formula (2.10), are given by the modification of equation (2.30) due to the application of Coriolis and centrifugal forces and an external conservative field. Hence H_Λ' takes the following form:

$$H_\Lambda' = \frac{1}{2}\int_\Lambda dx(i\nabla\psi^\star(x) - A(x)\psi^\star(x))(-i\nabla\psi(x) - A(x)\psi(x))$$
$$+ \int_\Lambda dx\, U(x)\psi^\star(x)\psi(x) + \int_{\Lambda^2} dx\, dy\, \psi^\star(x)\psi^\star(y)V(x-y)\psi(y)\psi(x), \quad (4.1)$$

where

$$A(x) = \omega \times x, \tag{4.2}$$

ω is a constant angular velocity directed along Oz and the centrifugal potential is absorbed into $U(x)$. It will be assumed that the potentials U and V are both lower bounded and invariant under rotations about Oz. The latter demand implies that these rotations form a dynamical symmetry group of the model. We define the representation, ρ, of this group in $Aut(\mathcal{A})$ by the following formula, expressed in terms of the cylindrical coordinates (r, ϑ, z) of an arbitrary point x,

$$\rho(\vartheta)\psi(r, \vartheta', z) = \psi(r, \vartheta + \vartheta', z)\,\forall\,\vartheta \in [0, 2\pi]. \tag{4.3}$$

Then the axial symmetry condition for the state ϕ is that

$$\phi \circ \rho(\vartheta) = \phi\,\forall\,\vartheta \in [0, 2\pi]. \tag{4.4}$$

Assume now that ϕ satisfies both the ODLRO and the axial symmetry conditions. Then, it follows from equations (2.22), (4.3) and (4.4) that, if Ψ serves as a macroscopic wavefunction

for ϕ, then so too does $\rho(\vartheta)\Psi$. Therefore since, by the remark following equation (2.22), the ODLRO condition defines Ψ up to a constant phase factor, it follows from equation (4.3) that

$$\rho(\vartheta)\Psi(r, \vartheta, z) = \kappa(\vartheta)\Psi(r, \vartheta, z), \tag{4.5}$$

where $\kappa(\vartheta)$ is a scalar of unit modulus. Further, since, by equation (4.3),

$$\rho(\vartheta)\rho(\vartheta') = \rho(\vartheta + \vartheta') \qquad \text{and} \qquad \rho(0) = I,$$

it follows from equation (4.5) that

$$\kappa(\vartheta)\kappa(\vartheta') = \kappa(\vartheta + \vartheta') \qquad \text{and} \qquad \kappa(0) = 1,$$

which signifies that κ is a one-dimensional representation of the circle, i.e. that

$$\kappa(\vartheta) = \exp(in\vartheta),$$

for some integer n, and consequently, by equation (4.5), that Ψ takes the form

$$\Psi(r, \vartheta, z) = f(r, z)\exp(in\vartheta). \tag{4.6}$$

Superselection sectors. It follows immediately from the above specifications that the quantum number n specifies a superselection sector, in that the normal folia of states with different values of this parameter are mutually disjoint. The integral character of n represents the Onsager–Feynman [15, 23] quantization rule.

Equilibrium and metastable states. Assume now that, under the prevailing thermodynamic conditions, the equilibrium state is unique and that it satisfies the ODLRO and axial symmetry conditions. Then it follows immediately that the quantum number n is zero when $\omega = 0$ for the following reasons. If $\omega = 0$, it follows from equations (4.1) and (4.2) that H'_Λ is invariant under the time reversal antiautomorphism τ, which sends $\psi(x)$ to $\psi^\star(x)$. Hence, by equations (2.10) and (2.19), the corresponding equilibrium state is τ-invariant and therefore, if Ψ serves as the macroscopic wavefunction for this state then, by the ODLRO condition (2.22), so too does $\overline{\Psi}$. In view of the remark following equation (2.22), this implies that Ψ and its complex conjugate differ only by a constant phase factor. Consequently, it follows from equation (4.6) that the winding number n must be zero in this case. This argument may easily be generalized to show that the winding number, n, of an equilibrium state is reversed if ω is replaced by $-\omega$ in equation (4.1), i.e. that, in an obvious notation,

$$n_\omega = -n_{-\omega}. \tag{4.7}$$

The question now arises whether n remains zero for some non-zero values of ω: if so we would have a kind of London rigidity against rotations [1]. This would manifest itself if n remained zero either

(a) in the equilibrium states corresponding to a certain range of values of ω; or

(b) in nonequilibrium metastable states that are stabilized by the above superselection rule, as in this case of persistent currents in superconducting rings (cf [16]).

One can readily envisage either possibility arising through a combination of the property (4.7) and the integral character of the winding number.

Superfluidity of condensate Assume now that the quantum number n is zero for the state ϕ, whether this is an equilibrium or a metastable state. Then, in this case, it follows from equation (4.6) that

$$\Psi(x) = f(r, z). \tag{4.8}$$

In order to see the hydrodynamical consequences of this formula, we note that the presence of the vector potential A in equation (4.1) for the formal Hamiltonian leads to an additional term $-A\psi^\star\psi$ in the formula for the current density observable in equation (2.9)''. Correspondingly it leads to a modification of formula (2.24) for the condensate current density by the addition of the term $-A|\Psi|^2$, while leaving equation (2.23) for the condensate density unchanged. Thus, using equation (4.2), we see that formula (2.24) is changed to

$$j_c(x) = -\frac{i}{2}(\overline{\Psi}(x)\nabla\Psi(x) - \Psi(x)\nabla\overline{\Psi}(x)) - |\Psi(x)|^2(\omega \times x). \tag{4.9}$$

Hence, by equations (2.23), (4.8) and (4.9), the transverse component of the condensate current is

$$j_c^{tr}(x) = -\rho_c(x)(\omega \times x) = -|f(r,z)|^2(\omega \times x). \tag{4.10}$$

This signifies that the transverse component of the condensate drift velocity, as viewed in the rotating frame, is $-\omega \times x$ and therefore that it is zero from the standpoint of an observer in the rest frame. This is just the manifestation of rotational superfluidity, wherein the condensate remains at rest while the container rotates [2]. Here we remark that this interpretation is dependent on the assumption of section 2 (following equation (2.27)) that the condensate and superfluid drift velocities are the same.

Comment. The general argument presented here is supported by results on rotational superfluidity obtained by Lewis and Pule [51] for the ideal Bose gas below its transition temperature and by Lieb and Seiringer [52] for the dilute interacting Bose gas at zero temperature. Note that, in the latter work, spontaneous symmetry breakdown is related to the appearance of two or more vortices. These are massive, i.e. not gapless, excitations. Indeed, the proofs of Goldstone's theorem in [44, 45] are not applicable here. The latter depends on the assumption of translational invariance and the former relies on an unproved assumption of a certain unproved asymptotic abelian property of the local observables with respect to space-time translations.

5. Translational superfluidity and the local instability of current-carrying states

Translational superfluidity corresponds typically to frictionless flow along a pipe and may be represented by translationally invariant current-carrying states of the model Σ in the idealization where the boundaries of the pipe are at infinity. Evidently, such states cannot enjoy global thermodynamical stability, since it follows from Galilei covariance that their free energy densities may be reduced by boosts opposing their drift velocities. It follows that the observed frictionless flow of superfluids must be carried by metastable, rather than globally stable, states. One may reasonably ask whether their metastability amounts to thermodynamical stability against modifications of state that are confined to bounded spatial regions, granted that there are models of other systems that support metastable states characterized by such local stability [39]. However, as we shall now show, translationally invariant current-carrying states of the present model, Σ, do not enjoy that kind of stability and therefore a different picture of their metastability is required. Such a picture will be proposed in section 6.

The objective of this section, then, is to show that a translationally invariant, current-carrying, locally normal state, ϕ, of the model Σ cannot be energetically stable against strictly localized modifications: in other words, for any such state ϕ, there is another state ϕ' that coincides with ϕ outside some bounded spatial region Λ and whose energy is lower than that of ϕ. This signifies that ϕ cannot be locally thermodynamically stable (LTS), in the sense defined in [38, 39], at zero temperature.

5.1. The strategy

We assume that ϕ is a translationally invariant, locally normal state of the model Σ of section 2, and that its drift velocity is $v(\neq 0)$. In order to construct a local modification of this state that lowers its energy, we start by introducing spheres $\Gamma, \Gamma_1, \Gamma_2$ and Γ_3 that are centred at the origin and whose radii are $R, (R + a), (R + b), (R + c)$, respectively, with $c > b > a > 0$. Here, a, b, c are fixed lengths, whereas R is a variable parameter, which may be made arbitrarily large. Thus, the regions between these spheres may be regarded as 'shells' and, of these, we denote $\Gamma_3 \backslash \Gamma$ and $\Gamma_2 \backslash \Gamma_1$ by Θ and Θ_1, respectively. We construct a linear, identity-preserving transformation τ of $\mathcal{A}$, that is completely positive (CP) in the sense defined by Stinespring [53], and an automorphism χ of this algebra according to the following specifications. τ reduces to the identity map outside the shell Θ and, within this region, its action on the position-dependent number density observable $n(x) := \psi^\star(x)\psi(x)$ serves to multiply it by a factor $g(x)^2$, where g is a smooth function of X into $[0, 1]$ that vanishes in Θ_1. This action therefore serves to evacuate the shell Θ_1. On the other hand, χ is a *local* gauge automorphism of $\mathcal{A}$ which rephases the field operator $\psi(x)$ by a factor $\exp(ih(x))$, where h is a smoothly varying function of position that vanishes outside Γ_2 and takes the form $-v \cdot x$ in Γ_1. Thus, χ induces a position-dependent boost, whose velocity is $-v$ in Γ_1 and zero outside Γ_2. The form of h, and thus of the boost velocity ∇h, in Θ_1 is irrelevant for our purposes, in view of the evacuation of this shell.

We define

$$\phi' := \chi^\star \tau^\star \phi, \tag{5.1}$$

where $\tau^\star$ and $\chi^\star$ are the duals of τ and χ, respectively. Thus, by the above specifications, the action of $\tau^\star$ on ϕ evacuates the shell Θ_1 and the subsequent action of $\chi^\star$ neutralizes the drift in Γ_1. We show that, under very general conditions on ϕ and the two-body potential V, the net effect of these transformations then leads to an energy decrease of order R^d in Γ and an energy change of order R^{d-1} in the shell Θ. Hence, for sufficiently large R, the transition $\phi \to \phi'$ leads to a net decrease of energy. This establishes that ϕ is not locally stable at zero temperature.

5.2. Constructions

We construct the CP map τ and the automorphism χ as locally normal transformations of $\mathcal{A}$. In accordance with the above specifications, we define χ to be the restriction to $\mathcal{A}$ of the automorphism of $\mathcal{F}$ denoted by the same symbol and formally defined by the equation

$$\chi \psi(x) = \psi(x) \exp(ih(x)), \tag{5.2}$$

where h is a smooth, real-valued function on X that vanishes outside Γ_2 and takes the form $h(x) = -v \cdot x$ in Γ_1.

We then define τ to be restricted to $\mathcal{A}$ of the composite of an isomorphism, ι, of $\mathcal{F}$ into $\mathcal{F} \otimes \mathcal{F}$ and a projection, π, of this product algebra onto $\mathcal{A}$, i.e.

$$\tau := (\pi \circ \iota)_{|\mathcal{A}}, \tag{5.3}$$

where π and ι are defined by the formulae

$$\pi(F \otimes F_0) = \phi_0(F_0)(pF) \,\forall\, F, \qquad F_0 \in \mathcal{F} \tag{5.4}$$

and

$$\iota\psi(x) = g(x)\psi(x) \otimes I + I \otimes (1 - g(x)^2)^{1/2}\psi(x). \tag{5.5}$$

Here p is the projection of $\mathcal{F}$ onto $\mathcal{A}$, defined in equation (2.18), ϕ_0 is the Fock vacuum state defined by equation (2.2) and g is a smooth, real-valued function on X such that

(a) $1 \geqslant g(x) \geqslant 0 \, \forall \, x \in X$;
(b) g vanishes in the shell Θ_1;
(c) g takes the value unity in both Γ and $X \backslash \Gamma_3$;
(d) $|\nabla g|$ has a finite, R-independent upper bound.

It follows easily from these specifications that τ is indeed a completely positive contractive transformation of $\mathcal{A}$ that reduces to the identity map outside the region $\Gamma \cup (X \backslash \Gamma_3)$.

The energy increment $E(\phi|\phi')$ This is the increment in the energy of Σ for a transition from ϕ to ϕ'. We formulate it in terms of the local energy observable H_Λ given by equation (2.12), which we rewrite in the following form:

$$H_\Lambda = \int_\Lambda \mathrm{d}x \, t(x) + \int_{\Lambda^2} \mathrm{d}x \, \mathrm{d}y \, n^{(2)}(x, y) V(x - y), \tag{5.6}$$

where $t(x)$ is the kinetic energy density defined in equation (**??**) and $n^{(2)}(x, y)$ is the pair density given by the formula

$$n^{(2)}(x, y) = \psi^\star(x)\psi^\star(y)\psi(y)\psi(x). \tag{5.7}$$

We shall assume that the two-body potential V is positive and measurable.

The energy increment $\Delta E(\phi|\phi')$ is defined (cf [38, 39]) by the formula

$$\Delta E(\phi|\phi') = \lim_{\Lambda \uparrow X} [\phi'(H_\Lambda) - \phi(H_\Lambda)]. \tag{5.8}$$

Now, by equations (5.1), (5.6) and (5.7),

$$\phi'(H_\Lambda) - \phi(H_\Lambda) = \int_\Lambda \mathrm{d}x \, \langle \phi; (\tau \chi - I)t(x) \rangle$$
$$+ \int_{\Lambda^2} \mathrm{d}x \, \mathrm{d}y \, V(x - y) \langle \phi; (\tau \chi - I)n^{(2)}(x, y) \rangle. \tag{5.9}$$

Further, it follows from equations (2.2), (2.9)$''$, (5.2)–(5.5) and (5.7), together with the fact that ψ annihilates the Fock vacuum vector Φ_0, that

$$\tau \chi t(x) = g(x)^2(t(x) + j(x) \cdot \nabla h(x) + \tfrac{1}{2}n(x)[\nabla h(x)]^2)$$
$$+ \tfrac{1}{2}[\nabla g(x)]^2 n(x) + \tfrac{1}{2}g(x)\nabla g(x) \cdot \nabla n(x) \tag{5.10}$$

and

$$\tau \chi n^{(2)}(x, y) = g(x)^2 g(y)^2 n^{(2)}(x, y). \tag{5.11}$$

Moreover, under the assumption that ϕ is a translationally invariant state carrying a current of drift velocity v, the expectation values of $n(x)$, $j(x)$ and $t(x)$ for this state are constants $\overline{n}$, $\overline{n}v$ and $\overline{t}$, respectively, which we assume to be finite; while that of $n^{(2)}(x, y)$ is a non-negative valued function $\overline{n}^{(2)}(x - y)$ of the difference of its arguments. We shall assume that this latter function is measurable. It follows then from equations (5.9)–(5.11) that

$$\phi'(H_\Lambda) - \phi(H_\Lambda) = \int_\Lambda \mathrm{d}x \left[\overline{t}(g(x)^2 - 1) + \overline{n}g(x)^2 (v \cdot \nabla h(x) + \frac{1}{2}(\nabla h(x))^2) + \frac{1}{2}\overline{n}(\nabla g(x))^2 \right]$$
$$+ \int_{\Lambda^2} \mathrm{d}x \, \mathrm{d}y (g(x)^2 g(y)^2 - 1)\overline{n}^{(2)}(x - y) V(x - y).$$

Recalling now that the restrictions of g to $\Gamma \cup (X \backslash \Gamma_3)$ and Θ_1 are 1 and 0, respectively, and that those of h to Γ_1 and $X \backslash \Gamma_2$ are $-v \cdot x$ and 0, respectively, it follows from this last equation that, for sufficiently large Λ,

$$\phi'(H_\Lambda) - \phi(H_\Lambda) = -\frac{1}{2}\overline{n}v^2 \int_{\Gamma_1} \mathrm{d}x \, g(x)^2 + \int_\Theta \mathrm{d}x \left[\overline{t}(g(x)^2 - 1) + \frac{1}{2}\overline{n}(\nabla g(x))^2 \right]$$
$$+ \int_{\Lambda^2} \mathrm{d}x \, \mathrm{d}y [g(x)^2 g(y)^2 - 1]\overline{n}^{(2)}(x - y) V(x - y)$$

Hence, by equation (5.8) and the positivity and measurability of $\overline{n}^{(2)}$ and V,

$$\Delta E(\phi|\phi') = -\frac{1}{2}\overline{n}v^2 \int_{\Gamma_1} \mathrm{d}x\, g(x)^2 + \int_{\Theta} \mathrm{d}x\left[\overline{t}(g(x)^2 - 1) + \frac{1}{2}\overline{n}(\nabla g(x))^2\right]$$

$$+ \int_{X^2} \mathrm{d}x\, \mathrm{d}y[g(x)^2 g(y)^2 - 1]\overline{n}^{(2)}(x - y)V(x - y). \tag{5.12}$$

Theorem 5.1. *Under the above assumptions, $\Delta E(\phi|\phi')$ is negative for sufficiently large R. Hence, as ϕ' coincides with ϕ outside the bounded region Γ_3, the latter state is unstable against some local modifications.*

Proof. Since g takes the value 1 in Γ and lies in the range [0, 1] in the shell $\Gamma_1\backslash\Gamma$, the first term on the rhs of equation (5.12) is negative and $O(R^d)$. On the other hand, as the volume of Θ is $O(R^{d-1})$, it follows from our specifications of g and h that the second term is $O(R^{d-1})$, while the third term is negative. This establishes that $\Delta E(\phi|\phi') < 0$ for R sufficiently large. $\qquad \square$

Comments

(1) This theorem does not conflict with Landau's picture of superfluidity [12], which requires the stability of a current-carrying state against the excitation of certain quasi-particles (phonons and rotons), while making no explicit demand of its stability against all local modifications.

(2) The theorem might seem to indicate that ϕ is not a ground state, in the strict sense specified in section 2. However, attempts to prove this point are impeded by problems connected with the domains of unbounded operators of the model.

(3) Similarly, such problems impede attempts to prove that current-carrying, translationally invariant states at finite temperatures cannot be thermal equilibrium states.

(4) On the other hand, in the case of lattice models, the argument employed to prove theorem 5.1 has been extended [39] to show that translationally invariant, current-carrying states cannot satisfy the equilibrium conditions either at zero or non-zero temperature.

6. Generalized Landau states

6.1. Stability of current-carrying states against elementary excitations

We have seen in the previous section that current-carrying states cannot be locally thermodynamically stable (theorem 5.1). We are thus led to look for a weaker stability condition which is able to account for their observed metastability. The chosen condition pertains to elementary excitations, which, as we shall see, comprise a mathematically precise modification of Landau's quasi-particle picture. The basic heuristic idea behind this choice is that the interaction of the system Σ with its environment is presumably of the few-particle type and thus the only kinetic processes it may be expected to generate are ones involving the creation of but a few elementary excitations, rather than the highly complicated ones considered in section 5.

This section is devoted to a treatment of the stability of current-carrying states against the elementary excitations. We start by introducing the Galilei boosted version of a ground state ϕ, namely

$$\phi_v = \phi \circ \xi(v), \tag{6.1}$$

where $\xi(v)$ is the boost given by equation (2.7); and we term ϕ_v a *generalized Landau state* if it is stable against elementary excitations, in a sense that will be made precise in due course (in definition 6.1).

Assuming the uniqueness of the ground state, ϕ and ϕ_v may be expressed as limits of their finite system counterparts in the following way. We define Λ_L to be the periodicized cube whose centroid is the origin and whose sides are parallel to the coordinate axes and are of length L; and we define $H_{N,L}$ to be the Hamiltonian of the N-particle version, $\Sigma_{N,L}$, of Σ that lives in this cube. Thus, $H_{N,L}$ is an operator in the N-particle subspace, $\mathcal{H}_{N,L}$, of the Fock space, $\mathcal{H}_L$, over Λ_L. We denote by $\Phi_{N,L}$ its ground state vector. This is then a vector in $\mathcal{H}_{N,L}$, and thus in $\mathcal{H}_L$, and hence the ground state $\phi_{N,L} := (\Phi_{N,L}, \cdot \Phi_{N,L})$ of $\Sigma_{N,L}$ extends to the gauge invariant subalgebra, $\mathcal{A}_L$, of $\mathcal{B}(\mathcal{H}_L)$ according to the formula

$$\phi_{N,L}(A) = (\Phi_{N,L}, A\Phi_{N,L}) \, \forall \, A \in \mathcal{A}_L.$$

Note that this state annihilates the observables whose particle numbers are different from N. Further, the dynamics of $\Sigma_{N,L}$ is given by the automorphisms $\alpha_{N,L}(t)$ of $\mathcal{A}_L$ given by the formula

$$\alpha_{N,L}(t)(A) = \exp(\mathrm{i}H_{N,L}t)A \exp(-\mathrm{i}H_{N,L}t). \tag{6.2}$$

Now the limits that concern us are those in which N and L tend to infinity in such a way that N/L^d is fixed at a finite value ρ. We assume that this limiting procedure yields the ground state ϕ of Σ and the dynamical automorphism group[7] α of this system, according to the formulae

$$\phi(A) = \lim_{L\to\infty; N=\rho L^d} \phi_{N,L}(A) \tag{6.3}$$

and

$$\phi(A[\alpha(t)B]C) = \lim_{L\to\infty; N=\rho L^d} \phi_{N,L}(A[\alpha_{N,L}(t)B]C) \tag{6.4}$$

for all local bounded observables A, B, C. It is clear that the above formulae imply that ϕ does indeed inherit the ground state property of $\phi_{N,L}$, since the condition for this is that the support of the distribution-valued Fourier transform of the function $t(\in \mathbf{R}) \to \phi(A\alpha_t B)$ lies in $\mathbf{R}_+$: as noted in section 2, this characterization of ground states is equivalent to the positivity condition on the Hamiltonian operator H of equation (2.15).

Turning now to space translations and Galilei boosts of $\Sigma_{N,L}$, we note that, in view of the periodicity of Λ_L, these are represented by the automorphisms $\sigma_{N,L}(x)$ and $\xi_{N,L}(v_L)$ of $\mathcal{A}_L$, respectively, given by the natural counterparts of equations (2.6) and (2.7), though with the components of v_L restricted to integral multiples of $2\pi/L$. Thus, the boosted ground state of $\Sigma_{N,L}$ is $\phi_{N,L} \circ \xi_{N,L}(v_L)$. To relate this to the boosted ground state ϕ_v of Σ, we assume that

$$\lim_{L\to\infty; N=\rho L^d : v_L \to v} \phi_{N,L}(\xi_{N,L}(v_L)A) = \phi_v(A) \tag{6.5}$$

for all local observables A. Hence, assuming Galilei covariance, the dynamical group associated with ϕ_v corresponds to a finite region Hamiltonian given by

$$H_{N,L}^v = H_{N,L} + v_L \cdot P_{N,L}, \tag{6.6}$$

where $P_{N,L}$ is the momentum observable of $\Sigma_{N,L}$. Here we have subtracted from the Hamiltonian the constant term $Nv_L^2/2$, which does not change the dynamical automorphism group. We remark that, in contrast with the situation expected, and sometimes proved, in relativistic quantum field theory [54], $H_{N,L}^v$ is not a positive operator. Further, the space

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[7] Here, for notational simplicity, we are taking this group to act on the observables rather than their GNS representation, as in the general scheme of section 2. However, the unemployment of the latter scheme yields the same results.

translational automorphisms $\sigma_{N,L}$ are implemented by a unitary group, whose generator is i times the momentum operator $P_{N,L}$, i.e

$$\sigma_{N,L}(x)A = \exp(iP_{N,L} \cdot x)A \exp(-iP_{N,L} \cdot x). \tag{6.7}$$

Assuming $\phi_{N,L}$ to be translationally invariant, it then follows that $P_{N,L}\Phi_{N,L} = 0$.

We now introduce the elementary excitations of $\Sigma_{N,L}$ and then pass to those of Σ in the following way. We assume that each of the elementary excitations of $\Sigma_{N,L}$ has a well-defined momentum k and energy $\varepsilon(k)$ in the following sense. For each finite set $(k_1, \ldots, k_m)$ in $\mathbf{R}^d$, with $m = O(1)$ w.r.t. N, there is a well-defined simultaneous eigenvector $\Phi_{N,L;k_1,\ldots,k_m}$, of $H_{N,L}$ and $P_{N,L}$, such that

$$P_{N,L}\Phi_{N,L;K_1,\ldots,K_M} = (k_1 + +k_m)\Phi_{N,L;k_1,\ldots,k_m} \tag{6.8}$$

and

$$H_{N,L}\Phi_{N,L;k_1,\ldots,k_m} = E_{N,L:k_1,\ldots,k_m}\Phi_{N,L;k_1,\ldots,k_m}, \tag{6.9a}$$

with

$$E_{N,L;k_1,\ldots,k_m} = E_{N,L;G} + \varepsilon_L(k_1) + \cdots + \varepsilon_L(k_m) + O(N^{-1}), \tag{6.9b}$$

where $E_{N,L;G}$ is the ground-state energy of $\Sigma_{N,L}$. This assumption is an abstraction of Lieb's results [49] on the Lieb–Liniger model [24]. There two branches of elementary excitations arise, and a choice must be made [49]. The term $O(N^{-1})$ is due to interactions between the elementary excitations: the latter occurs also in the case of the spin-waves in the Heisenberg model [55].

As a first step to passing from the excitations of $\Sigma_{N,L}$ to those of Σ, we introduce the Schwartz space $\mathcal{D}(X^m)$ and define

$$\Phi_{N,L;f} = (2\pi/L)^{md/2}\sum\nolimits_{k_1,\ldots,k_m} f(k_1, \ldots, k_m)\Phi_{N,L;k_1,\ldots,k_m} \, \forall f \in \mathcal{D}(X^m), \tag{6.10}$$

where the k's run over the integral multiples of $2\pi/L$. The factor $(2\pi/L)^{md/2}$ ensures that $\|\Phi_{N,L;f}\|$ reduces to a finite quantity, namely $\|f\|_{L^2(X)}$, in the limit $N \to \infty$. We assume that the inner products of $\Phi_{N,L;f}$ with $[\alpha_{N,L}(t)A]\Phi_{N,L}$, $[\sigma_{N,L}(x)A]\Phi_{N,L}$ and $[\xi_{N,L}(v_L)A]\Phi_{N,L}$ converge to canonical counterparts for the system Σ as $N, L \to \infty$, $v_L \to v$ and $N/L^d = \rho$. Thus, they serve to define a vector Φ_f in the representation space $\mathcal{H}$ of Σ with the properties that

$$(\Phi_f, [\alpha_t A]\Phi) = \lim_{L\to\infty; N=\rho L^d}(\Phi_{N,L;f}, \alpha_{N,L}(t)A]\Phi), \tag{6.11}$$

$$(\Phi_f, [\sigma(x)A]\Phi) = \lim_{L\to\infty; N=\rho L^d}(\Phi_{N,L;f}, \sigma_{N,L}(x)A]\Phi) \tag{6.12}$$

and

$$(\Phi_f, [\xi(v)A]\Phi) = \lim_{L\to\infty; N=\rho L^d; v_L\to v}(\Phi_{N,L;f}, \xi_{N,L}(v_L)A]\Phi). \tag{6.13}$$

The vector Φ_f may now be unsmeared and expressed in terms of a vector-valued distribution $\Phi_{k_1,\ldots,k_m}$ according to the formula

$$\Phi_f = \int_{X^m} dk_1 \cdots dk_m f(k_1, \ldots, k_m)\Phi_{k_1,\ldots,k_m}. \tag{6.14}$$

It now follows from our specifications, especially the unitary implementations of the space and time translational automorphisms defined by equations (2.14) and (2.16), that

$$U(t)\Phi_{k_1,\ldots,k_m} = \exp(i(\varepsilon(k_1) + \cdots + \varepsilon(k_m))t)\Phi_{k_1,\ldots,k_m} \tag{6.15}$$

and

$$S(x)\Phi_{k_1,\dots,k_m} = \exp(\mathrm{i}(k_1 + \cdots + k_m) \cdot x)\Phi_{k_1,\dots,k_m}. \tag{6.16}$$

Hence, by equations (2.15) and (2.17), $\Phi_{k_1,\dots,k_m}$ is a simultaneous eigenvector of the Hamiltonian, H, and momentum operator, P, such that

$$H\Phi_{k_1,\dots,k_m} = (\varepsilon(k_1) + \cdots + \varepsilon(k_m))\Phi_{k_1,\dots,k_m} \tag{6.17}$$

and

$$P\Phi_{k_1,\dots,k_m} = (k_1 + \cdots + k_m)\Phi_{k_1,\dots,k_m}. \tag{6.18}$$

These last two formulae represent the infinite volume limits of equations (6.8) and (6.9).

At this stage we are able to state what we mean by the generalized Landau state.

Definition 6.1. *We say that ϕ_v is a generalized Landau state if the following condition—the Landau superfluidity condition—holds: there exists $v_c > 0$ such that*

$$\varepsilon(k) + v \cdot k \geqslant 0 \,\forall\, k \in \mathbf{R}^d \qquad \text{if} \quad |v| \leqslant v_c. \tag{6.19}$$

Choosing k in the direction of $-v$ in equation (6.19), we arrive at the well-known explicit formula for v_c:

$$v_c = \inf_k(\varepsilon(k)/|k|) \tag{6.20}$$

By equations (6.6), (6.8), (6.9), (6.17) and (6.18), equation (6.19) represents the stability of the state ϕ_v against generation of a finite number of elementary excitations. It should be emphasized that Landau formulated condition (6.19) at a not entirely quantum level, but rather in a semi-classical hydrodynamical framework. Sometimes condition (6.19) is expressed, rather, in terms of quasi-particles. However, the meaning of this concept is somewhat vague, though Lieb [49] attempted to define it precisely as a pole of a Green's function. By contrast, the elementary excitations on which we base our considerations are exact eigenfunctions of the system satisfying certain well-defined conditions [49].

The sound velocity, v_s, is defined in terms of the elementary excitations by the formula

$$v_s = |(\lim_{k \to 0} \partial \varepsilon(k)/\partial k)|, \tag{6.21}$$

On the other hand, by a macroscopic argument [2], one expects that v_s is also given by

$$v_s = (\rho\kappa_0)^{-1/2}, \tag{6.22}$$

where κ_0 is the compressibility of the system at zero temperature. This is defined in terms of the thermodynamic limit, $e(\rho)$, of the energy density and the pressure $P(\rho)$ by the formulae

$$\kappa_0 = \left[\rho\frac{\mathrm{d}P}{\mathrm{d}\rho}\right]^{-1}, \tag{6.23}$$

where (cf [19, P 58])

$$P = \rho\frac{\mathrm{d}e(\rho)}{\mathrm{d}\rho} - e(\rho) \tag{6.24}$$

and

$$e(\rho) = \lim_{L \to \infty; N = \rho L^d} L^{-d} E_{N,L:G}. \tag{6.25}$$

Under quite general conditions on the pair interactions, it may be proved that [32]

$$\kappa_0 \geqslant 0. \tag{6.26}$$

Further it follows from equations (6.19)–(6.21) that

$$v_c \leqslant v_s > 0. \tag{6.27}$$

We remark that, in the case of liquid Helium II, heuristic, empirically supported arguments indicate that v_c is less than v_s and that its value is given by the slope of the line through the origin in the $\varepsilon - |k|$ plane that is tangent to the graph of ε in the neighbourhood of its local minimum (at $k \neq 0$), which is governed by the roton spectrum [50].

Clearly, a rigorous proof of the identity between the two expressions (6.20) and (6.21) may be expected to be highly nontrivial. We shall come back to this point in section 6.3, where we prove it in a self-contained manner for the Girardeau [25] model.

6.2. Stability condition for non-translationally covariant systems

The treatment of section 6.1 is only appropriate for *homogeneous* systems, in which momentum is conserved. The question may be posed, however, whether a natural replacement of the Landau superfluidity condition (definition 6.1) exists for inhomogeneous systems, such as the rotating system of section 4 and the LSY model of a dilute, trapped Bose gases [4, 18]. In this connection it is noteworthy that, according to the condition for translational superfluidity of the latter model, the free Bose gas is also a superfluid, while it is not so according to the Landau criterion (6.19). In this section we propose a replacement for the *necessity* part, represented by equation (6.27), of Landau's condition for inhomogeneous systems. Since that inequality, together with condition (6.22) signifies that the compressibility is finite, we proceed as follows.

Definition 6.2. *We take the necessity part of Landau's superfluidity condition, even for inhomogeneous systems, to be that the compressibility is finite, i.e. that*

$$\kappa_0 < \infty. \tag{6.28}$$

For non-translationally invariant systems, the elementary excitations are eigenfunctions of the Hamiltonian which do not carry a definite momentum, but show up in various quantities, such as the density of states and in the specific heat, displaying the characteristic phonon-behaviour $O(T^3)$ for low temperatures T. Since we are not able to formulate a stability condition in a mathematically precise sense as in section 6.1, we regard the word 'superfluidity' in definition 6.2 in the phenomenological sense which is briefly discussed in section 2, with $\rho_n = 0$ in equations (2.25)–(2.27) since we are referring to the ground state.

We remark that, in the case of the ideal Bose gas, condition (6.28) is violated at $T = 0$. It is not known rigorously as yet whether it holds there for dilute trapped Bose gases in the Gross–Pitaevski limit, because the presently known rigorous bounds [4] do not allow control over the compressibility.

We devote section 6.3 to the Girardeau model [25], a special case of the Lieb–Liniger model [24] (the forthcoming equation (6.29)) which seems to be the only model for which the elementary excitations may be derived without further assumptions [49] and, moreover, allows an elementary derivation of the thermodynamic limit of the compressibility. The purpose of that section is two-fold. On the one hand, the model allows an elementary proof of the identity between equations (6.21) and (6.22), which is used to motivate definition 6.2; on the other hand, it shows very clearly how the repulsive interaction introduces a special structure of the ground-state wavefunction (quite different from a product of one-particle wavefunctions in the same state, as in the free Bose gas), which complies with both definitions 6.1 and 6.2.

6.3. The Girardeau model and the Landau superfluidity condition

We start with the model of N particles in one dimension with (repulsive) delta function interactions [24], whose formal Hamiltonian is given by

$$H_{N,L} = -\frac{1}{2} \sum\nolimits_{i=1}^{N} \frac{\partial^2}{\partial x^2} + 2c \sum\nolimits_{i,j=1}^{\prime N} \delta(x_i - x_j), \qquad 0 \leqslant x_i, \quad x_j \leqslant L, \tag{6.29}$$

the prime over the second Σ indicating that summation is confined to nearest neighbours. For the (standard) rigorous definition corresponding to (6.29) see [56].

The limit, as $c \to \infty$, of equation (6.29) yields Girardeau's model [25]. This seems to be the only model for which both the (double) spectrum of elementary excitations and the compressibility has been obtained rigorously without (as yet) unproved assumptions. The former were obtained by Lieb [49] and although the result for the compressibility follows from specializing the result of the appendix to that article (made rigorous by Dorlas [56]) to the present case, the following derivation, which does not seem to be found in the literature, is elementary and provides a transparent physical reason for the validity of property (6.28).

In the limit $c \to \infty$ of equation (6.29), the boundary condition on the wavefunctions reduces to

$$\psi(x_1, \ldots, x_N) = 0 \qquad \text{if} \quad x_j = x_l, \quad 1 \leqslant j < l \leqslant N \tag{6.30}$$

and the (Bose) eigenfunctions satisfying equation (6.30) simplify to

$$\psi^B(x_1, \ldots, x_N) = \psi^F(x_1, \ldots, x_N) A(x_1, \ldots, x_N), \tag{6.31}$$

where ψ^F is the Fermi wavefunction for the free system of N particles confined to the region $0 \leqslant x_i < L, i = 1, \ldots, N$, with periodic boundary conditions, and

$$A(x_1, \ldots, x_N) = \prod\nolimits_{j<l} \operatorname{sgn}(x_j - x_l). \tag{6.32}$$

Note that ψ^F automatically satisfies equation (6.30) by the exclusion principle. Indicating the Bose and Fermi ground states by the subscript 0, it follows from equation (6.31) and the non-negativity of ψ_0^B that

$$\psi_0^B = |\psi_0^F|. \tag{6.33}$$

Since $A^2 = 1$, by equation (6.32), the correspondence between ψ^B and ψ^F given by equation (6.31) preserves all scalar products, and therefore the energy spectrum of the Bose system is the same as of the free Fermi gas Further, ψ_0^F is a Slater determinant of plane-wavefunctions labelled by wave vectors $k_i, i = 1, \ldots, N$, equally spaced over the range $[-k_F, k_F]$, where k_F is the Fermi momentum. Hence k_F is equal to $\pi(N-1)/L$, which in the thermodynamic limit reduces to

$$k_F = \pi\rho. \tag{6.34}$$

The simplest excitation is obtained by moving a particle from k_F to $q > k_F$ (or from $-k_F$ to $q < -k_F$), thereby leaving a hole at k_F (or $-k_F$). This excitation has momentum $p = (q - k_F)$ (or $-(q - k_F)$) and energy $\varepsilon(p) = (q^2 - k_F^2)/2$, i.e.

$$\varepsilon(p) = p^2/2 + k_F|p|. \tag{6.35}$$

It follows from this formula and equations (6.21) and (6.35) that the sound velocity is

$$v_s = k_F = \pi\rho. \tag{6.36}$$

On the other hand, the ground-state energy density is given by the following standard formula for that of a one-dimensional ideal spinless Fermi gas in the thermodynamical limit (cf [57, section 56]).

$$e(\rho) = (2\pi)^{-1} \int_{-k_F}^{k_F} dk \, k^2/2,$$

i.e. by equation (6.34),

$$e(\rho) = \frac{\pi^2}{6}\rho^3 \tag{6.37}$$

and consequently, by equations (6.23) and (6.24),

$$P = \frac{\pi^2}{3}\rho^3 \tag{6.38}$$

and

$$\kappa_0^{-1} = \pi^2\rho^3. \tag{6.39}$$

According to this formula and equation (6.22), v_s is given by equation (6.36), which confirms that, for Girardeau's model, the phenomenological formulae (6.22), (6.23) yield the same result as the quantum mechanical one, (6.21).

We note that here the role of the repulsive interaction in leading to the Fermi distribution of momenta and thus to the finite compressibility. Equivalently, the structure, given by equation (6.33), of the ground-state wavefunction leads to a non-zero range of values of the particle momenta and hence to a non-zero velocity of sound. By contrast, for an ideal Bose gas, the particle momenta would all be concentrated at the value zero and, as a consequence, the speed of sound would be zero.

In spite of its apparent simplicity, the above model is far from trivial. Some insight into its complexities is obtained when looking at correlation functions: there exists a result due to Lenard [58] on the Fourier transform of the one-particle density matrix, implying the absence of ODLRO in the model, and just a few results on higher order correlations [59], restricted, however, to Dirichlet and Neumann boundary conditions.

We conclude that the Landau condition depends very strongly on the specific structure of the ground-state wavefunction, in contrast to London rigidity, which depends only on very general properties of the system, such as ODLRO. The latter is, however, not necessary for translational superfluidity in the Landau picture as shown, again, by the Girardeau model, bearing in mind Lenard's above cited result.

7. Conclusion

In this paper we have analysed two aspects of the mathematical theory of superfluidity. These are centred on the ODLRO condition and on a precise form of Landau's condition for the stability of uniform currents against elementary excitations.

In fact, the condition of ODLRO corresponds to that of gauge symmetry breakdown for the description of the bosonic model in terms of its field, rather than observable, algebra. Thus, the equilibrium states with ODLRO enjoy the properties of both gauge and Galilei symmetry breakdown and the resultant emergence of Goldstone bosons and long range correlations are described in section 3. Further, as shown in section 4, the combination of ODLRO and axial symmetry leads to a 'rotational superfluidity' of the system, which is manifested when the system is placed in a rotation bucket. This kind of superfluidity stems a superselection rule that represents a macroscopic form of London rigidity. It is found to prevail in both the ideal Bose gas [51] and in the LSY model of a dilute, trapped interacting Bose gas when the angular velocity of rotation ω does not exceed a critical value ω_c. In the latter model, the Hamiltonian in the rotating frame becomes unbounded from below, due to an instability against the creation of an unlimited number of vortices, when $\omega > \omega_c$ [43].

We have proved in section 5 (theorem 5.1) that, under very general conditions, translationally invariant current-carrying states are not locally thermodynamically stable,

in the sense specified in [38, 39]. This implies that a weaker kind of stability is needed to characterize the observed metastability of current-carrying states. The picture of this metastability formulated in section 6 is expressed in terms of the concept of generalized Landau states (definitions 6.1 and 6.2), which are translationally invariant current-carrying ones that are stable against finite numbers of *elementary excitations*, which in general are not the same as quasi-particles. This picture is realized by the Lieb–Liniger model [24], under the assumption that the elementary excitations are given by the Bethe Ansatz. In the special case where this model reduces to that of Girardeau [25], no such supplementary assumption is needed, and our explict review of its properties demonstrates how this neo-Landau kind of superfluidity, by contrast with the rotational version, depends very strongly on the special structural properties of the ground state.

Finally, we would like to mention some open problems. The first is whether the necessity part of Landau's condition (definition 6.2) is realized by the model of dilute trapped gases [4]: the existing estimates do not, so far, allow control over the compressibility [4]. Another problem is whether theorem 5.1 can be extended to a proof that generalized Landau states do not satisfy the KMS or ground-state conditions: what is lacking there is a proof that the LTS and the KMS (or ground state) conditions are equivalent for continuous systems, as they have been proved to be [38, 39] for lattice models. Evidently, that is a major problem, which goes beyond the context of the theory of superfluidity. So too is the problem of whether ground states on the algebra of observables extend to those on the field algebra in the same way as KMS states have been proved [40] to do: a proof that they do so would establish a one-to-one correspondence between ODLRO and gauge symmetry breakdown and thereby complete the picture discussed in section 3.

Acknowledgments

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